2.3 Differentiation Rules

The derivative has several different types of notation:

Notation	Meaning
$f'(x)$	Derivative of $f(x)$
$\frac{dy}{dx}$	Derivative of y with respect to x.
y'	Derivative of y
$\frac{d}{dx}[f(x)]$	Derivative of f with respect to x.

Constant Rule

$\frac{d}{dx}[c]=0$ This means the derivative of any number is zero. For example, suppose we had $y=9$ and the question asked us to find y'. Then our answer would automatically be zero because y = 9 is a horizontal line. The slope of a horizontal line is always 0.

EXAMPLE: Find $f'(x)$ if $f(x)=4\cdot\pi\cdot e^2$.

Since $e=2.71...$ and $\pi=3.14...$ then this whole equation is a constant, so $f'(x)=0$.

Can you see a pattern of the derivative with the following table?

Original function	Derivative
$y=x^2$	$y'=2x$
$y=x^3$	$y'=3x^2$
$y=x^5$	$y'=5x^4$

It seems like the power comes down and the exponent is reduced by one. We can use this information to derive a formula for the derivative for $y=x^n$. This will be called the power rule.

Power Rule

If n is any real number, then $\frac{d}{dx}x^n=nx^{n-1}$ for all x where powers of x^n and x^{n-1} are defined.

Other Derivative Rules

$$\frac{d}{dx}[f(x)+g(x)]=f'(x)+g'(x)$$
$$\frac{d}{dx}[f(x)-g(x)]=f'(x)-g'(x)$$
$$\frac{d}{dx}[c\cdot f(x)]=c\cdot f'(x)$$

EXAMPLE: If $y = x^{12}$, find y'.

We will use the power rule formula. Here n = 12. $y' = 12x^{12-1}$ so $y' = 12x^{11}$.
EXAMPLE: If $y = 3x^4$, find y'.

We can write y as $y = 3 \cdot x^4$. Now we use the power rule: $y' = 3 \cdot 4x^3$, so $y' = 12x^3$

EXAMPLE: If $g(x) = \frac{3}{2}x^6 - x + 3$, find $g'(x)$.

First we will rewrite this as $g(x) = \frac{3}{2} \cdot x^6 - 1x^1 + 3x^0$. For this one we can apply the power rule to each term separately: $g'(x) = \frac{3}{2} \cdot 6x^5 - 1x^0 + 0$. After simplifying we get: $g'(x) = 9x^5 - 1$.

EXAMPLE: If $y = 3x(6x - 5x^2)$, find y'.

I recommend distributing first. You will get: $y = 18x^2 - 15x^3$. Now apply the power rule. You will get: $y' = 36x - 45x^2$. You can also write it as $y' = 9x(4 - 5x)$

EXAMPLE: If $f(x) = \sqrt[4]{x}$, find $f'(x)$.

This one can be rewritten with fractional exponents: $f(x) = x^{\frac{1}{4}}$. We still can apply the power rule here: $f(x) = \frac{1}{4}x^{\frac{1}{4}-1}$, so after simplifying you get $f(x) = \frac{1}{4}x^{-\frac{3}{4}}$. You do not want this as a negative exponent, so place x in the denominator. You will get $f(x) = \frac{1}{4x^{\frac{3}{4}}}$.

EXAMPLE: If $f(x) = \frac{2}{x^3}$, find $f'(x)$.

If there is a variable in the denominator, write this as a negative exponent. You will get $f(x) = 2x^{-3}$. Then apply the power rule. You will get $f'(x) = -6x^{-4}$. Now write it without negative exponents and you will have $f'(x) = -\frac{6}{x^4}$.
EXAMPLE: If $f(x) = x + \frac{1}{x}$, find $f'(x)$.

We need to first rewrite this as a negative exponent. You will get $f(x) = x + x^{-1}$. Apply the power rule. You will get $f'(x) = 1 - x^{-2}$. Now rewrite without negative exponents: $f'(x) = 1 - \frac{1}{x^2}$.

EXAMPLE: If $f(x)=\dfrac{2x^2-3x+1}{x}$, find $f'(x)$.

I would suggest dividing each term by x and then simplifying: $f(x)=\dfrac{2x^2}{x}-\dfrac{3x}{x}+\dfrac{1}{x}$ so $f(x)=2x-3+x^{-1}$.

Now apply the power rule: $f'(x)=2-x^{-2}$. Finally rewrite with positive exponents: $f'(x)=2-\dfrac{1}{x^2}$.

EXAMPLE: If $y=\sqrt{x}-\dfrac{5}{\sqrt{x}}+\sqrt{2}$, find y'.

First write change the radicals into fractional exponents: $y=x^{\frac{1}{2}}-5x^{-\frac{1}{2}}+\sqrt{2}$. I did not change the last square root of 2 into a fractional exponent because when I take the derivative this term will turn to zero. Now apply the product rule: $y'=\dfrac{1}{2}x^{-\frac{1}{2}}+\dfrac{5}{2}x^{-\frac{3}{2}}$. Lastly rewrite with positive exponents: $y'=\dfrac{1}{2x^{\frac{1}{2}}}+\dfrac{5}{2x^{\frac{3}{2}}}$. You may leave it with fraction exponents unless the question specifically tells you not to. Suppose this question did want you to write it with radicals. Then you would have: $y'=\dfrac{1}{2\sqrt{x}}+\dfrac{5}{2\sqrt{x^3}}$, which is also $y'=\dfrac{1}{2\sqrt{x}}+\dfrac{5}{2x\sqrt{x}}$.

With common denominators it can be written as $y'=\dfrac{x+5}{2x\sqrt{x}}$.

EXAMPLE: Find the equation of the tangent line to the curve $y=3x^3-6$ at the point (2, 18).

We will first find the derivative using the power rule: $y'=9x^2$. We can put in a 2 for x to get our slope: $m=9(2)^2=36$. Now we need to find the equation of a line with a slope of 36 that passes through (2, 18). We can use y = mx + b. Here m = 36, x = 2 and y = 18: Put it into the y = mx + b formula: 18 = 36(2) + b. So solving this we get -54 for b. The equation is y = 36x – 54.

EXAMPLE: Find the equation of the tangent line to the curve $y=3(5-x)^2$ at the point (5, 0).

To find the derivative we will first distribute: $y=3(25-10x+x^2)$ so $y=3x^2-30x+75$. Now we take the derivative: $y'=6x-30$. We find the slope by putting in a five for x: m = 6(5) – 30. So m = 0. This is a horizontal line going through (5, 0), so the equation is y = 0.

EXAMPLE: Determine the point(s) at which $y=x^2+1$ has a horizontal tangent line.

A horizontal tangent line means that the slope is zero. So we must take the derivative and set it equal to zero.

$y'=2x$	We have our derivative. Now we must set it equal to zero.
$0=2x$	Now we solve for x.
$x=0$	So when x = 0, y = 1, so the point is (0, 1).

EXAMPLE: Determine the point(s) at which $y = x^3 - 27x$ has a horizontal tangent line.

$y' = 3x^2 - 27$ We have our derivative. Now we must set it equal to zero.
$0 = 3x^2 - 27$ Now we solve for x.
$27 = 3x^2$
$9 = x^2$
$x = \pm 3$ Put each of these into the original equation for x and we get (3, -54) and (-3, 54).

We unfortunately can't use the product rule on everything. We will look at ways to find derivatives of things multiplied together and divided.

Product Rule

$$\frac{d}{dx}[f \cdot g] = f \cdot g' + g \cdot f'$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{g \cdot f' - f \cdot g'}{g^2}$$

EXAMPLE: Use the product rule to find $H'(x)$ if $H(x) = \underset{f}{(6x+5)}\underset{g}{(x^3 - 2)}$.

In our problem, I labeled f and g. Now we apply the product rule formula. I have labeled each part.

$$H'(x) = \underset{f}{(6x+5)}\underset{g'}{(3x^2)} + \underset{g}{(x^3 - 2)}\underset{f'}{(6)}$$ Now we just simplify.

$$H'(x) = 18x^3 + 15x^2 + 6x^3 - 12$$
$$H'(x) = 24x^3 + 15x^2 - 12$$

EXAMPLE: Use the quotient rule to find $H'(x)$ if $H(x) = \frac{x^2 + 2}{2x - 7}$.

Here the numerator is always f and the denominator is always g. Now we use the quotient rule:

$$H'(x) = \frac{\underset{g}{(2x-7)}\underset{f'}{(2x)} - \underset{f}{(x^2+2)}\underset{g'}{(2)}}{\underset{g^2}{(2x-7)^2}}$$ Now we distribute and simplify.

$$H'(x) = \frac{4x^2 - 14x - 2x^2 - 4}{(2x-7)^2}$$ Now just simplify to get our answer: $H'(x) = \frac{2x^2 - 14x - 4}{(2x-7)^2}$

EXAMPLE: Use the product rule to find $M'(x)$ if $M(x) = \underset{f}{\sqrt{x}}\underset{g}{\left(4 - x^2\right)}$. Write answer as a single fraction.

Once again f and g are labeled. We need to use the product rule again.

$$M'(x) = \underset{f}{\sqrt{x}}\underset{g'}{(-2x)} + \underset{g}{\left(4 - x^2\right)}\underset{f'}{\frac{1}{2}x^{-\frac{1}{2}}}$$ Now get rid of the negative exponent and rewrite with radicals.

$$M'(x) = -2x\sqrt{x} + \frac{4 - x^2}{2\sqrt{x}}$$ We need common denominators to write this as a single fraction.

$$M'(x) = \frac{-2x\sqrt{x}}{1} \cdot \left(\frac{2\sqrt{x}}{2\sqrt{x}}\right) + \frac{4 - x^2}{2\sqrt{x}}$$ We need to multiply across the top and across the bottom.

$$M'(x) = \frac{-4x^2}{2\sqrt{x}} + \frac{4 - x^2}{2\sqrt{x}}$$ Now combine as a single fraction.

$$M'(x) = \frac{4 - 5x^2}{2\sqrt{x}}$$

EXAMPLE: Use the product rule to find $M'(x)$ if $M(x) = \underset{f}{(x^2 + 1)}\underset{g}{\left(x + 5 + 1/x\right)}$.

$$M'(x) = \underset{f}{(x^2 + 1)}\underset{g'}{\left(1 - \frac{1}{x^2}\right)} + \underset{g}{\left(x + 5 + \frac{1}{x}\right)}\underset{f'}{(2x)}$$ Now multiply this together.

$$M'(x) = x^2 - 1 + 1 - \frac{1}{x^2} + 2x^2 + 10x + 2$$ We simplify and add like terms.

$$M'(x) = 3x^2 + 10x + 2 - \frac{1}{x^2}$$ We can leave our answer like this.

EXAMPLE: Use the quotient rule to find $H'(x)$ if $H(x) = \frac{x}{\sqrt{x} - 1}$.

We will use the quotient rule again. Remember the numerator is always f and the denominator is always g.

$$H'(x) = \frac{\left(\sqrt{x} - 1\right)(1) - x \cdot \frac{1}{2}x^{-\frac{1}{2}}}{\left(\sqrt{x} - 1\right)^2}$$ Now we will simplify. The x's cancel this way: $x \cdot x^{-\frac{1}{2}} = x^{\frac{1}{2}}$.

$$H'(x) = \frac{\sqrt{x} - 1 - \frac{1}{2}x^{\frac{1}{2}}}{\left(\sqrt{x} - 1\right)^2}$$ Now we will multiply the top and bottom by 2 to clear the fraction.

$$H'(x)=\frac{2}{2}\cdot\frac{\sqrt{x}-1-\frac{1}{2}x^{\frac{1}{2}}}{\left(\sqrt{x}-1\right)^2}$$

$$H'(x)=\frac{2\sqrt{x}-2-\sqrt{x}}{2\left(\sqrt{x}-1\right)^2}$$ I multiplied by 2 and then changed the fractional power back into a radical.

$$H'(x)=\frac{\sqrt{x}-2}{2\left(\sqrt{x}-1\right)^2}$$ I added the like terms on top and this is as far as we can go.

EXAMPLE: Use the quotient rule to find $B'(x)$ if $B(x)=x^4\left(1-\frac{2}{x+1}\right)$.

I want to first get common denominators inside the parenthesis before I start taking derivatives.

$$B(x)=x^4\left(\frac{x+1}{x+1}-\frac{2}{x+1}\right)$$ Now combine to get a single fraction inside the parenthesis.

$$B(x)=x^4\left(\frac{x+1-2}{x+1}\right)$$ Now simplify the top.

$$B(x)=x^4\left(\frac{x-1}{x+1}\right)$$ To avoid combining a product and quotient rule, distribute the x^4.

$$B(x)=\frac{x^5-x^4}{x+1}$$ Now we are ready to use the quotient rule.

$$B'(x)=\frac{(x+1)(5x^4-4x^3)-(x^5-x^4)(1)}{(x+1)^2}$$ We want to multiply and simplify the numerator.

$$B'(x)=\frac{5x^5-4x^4+5x^4-4x^3-x^5+x^4}{(x+1)^2}$$ I multiplied the first two terms and distributed the negative.

$$B'(x)=\frac{4x^5+2x^4-4x^3}{(x+1)^2}$$ You can leave your answer like this or you could also factor to get:

$$B'(x)=\frac{2x^3(2x^2+x-2)}{(x+1)^2}$$ This is as far as we can take it.

More on next page…

Higher Order Derivatives

$f(x)$	This is our original function
$f'(x)$	First derivative of f
$f''(x)$	Second derivative of f (derivative of $f'(x)$)
$f'''(x)$	Third derivative of f (derivative of $f''(x)$)
$f^{(n)}(x)$	The nth derivative of f (derivative of $f^{(n-1)}(x)$)

A function has derivatives of all orders once the derivative reaches zero.

EXAMPLE: Let $f(x) = 4x^3 + 5x^2 + 3x + 1$. Find the derivatives of all orders.

We will use the power rule for this.

$f'(x) = 12x^2 + 10x + 3$	In order to find $f''(x)$, take the derivative of $f'(x)$ using the power rule.
$f''(x) = 24x + 10$	Now we will take the derivative of $f''(x)$ to get $f'''(x)$.
$f'''(x) = 24$	
$f^{(4)}(x) = 0$	The derivative is zero, and all subsequent derivatives are zero, so we've found the derivatives of all orders.

EXAMPLE: Let $f(x) = \dfrac{x^3 + 2x^2 - 1}{x}$. Find $f'''(x)$.

You might think that we need to use the quotient rule for this one, but we don't need to. We can simplify this by dividing each term by x: $f(x) = \dfrac{x^3}{x} + \dfrac{2x^2}{x} - \dfrac{1}{x}$. This simplifies to $f(x) = x^2 + 2x - x^{-1}$

$f(x) = x^2 + 2x - x^{-1}$	
$f'(x) = 2x + 2 + x^{-2}$	Take the first derivative using the power rule.
$f''(x) = 2 - 2x^{-3}$	In order to find $f''(x)$, take the derivative of $f'(x)$ using the power rule.
$f'''(x) = 6x^{-4}$	Now we will take the derivative of $f''(x)$ to get $f'''(x)$.
$f'''(x) = \dfrac{6}{x^4}$	