4.5 Indeterminate Forms and L'Hopital's Rule

In this section we will be looking at limits again. Particularly we are looking at limits that result in indeterminate forms (expressions that cannot be calculated). In this case we can apply L'Hopital's Rule in order to evaluate limits that result in indeterminate forms.

Types of indeterminate forms: $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$

L'Hopital's Rule: Suppose that $f(a) = g(a) = 0$ and that f and g are differentiable on an open interval *I* containing *a*, and that $g'(x) \neq 0$ on I if $x \neq a$. Then:
$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, assuming that the limit on the right side of this equation exists.

Using L'Hopital's Rule:

To find $\lim_{x \to a} \frac{f(x)}{g(x)}$ by L'Hopital's Rule, keep taking the derivative of *f* and *g* as long as you keep ending up with the form 0/0 at $x = a$. As soon as one of these derivatives is not zero at $x = a$ then stop taking derivatives and find the limit. Remember the L'Hopital's rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x \to 3} \frac{x-3}{x^2-9}$. Then evaluate it using limit rules that we studied in Chapter 1.

If we plug in 3 for each x, we will get 0/0. This means we can apply L'Hopital's Rule. We will take the derivative of the top and bottom separately resulting in: $\lim_{x \to 3} \frac{1}{2x}$. If we plug in 3 for the top and bottom we will get 1/6. This means L'Hopital's Rule no longer applies, so we will stop taking derivatives. The answer is 1/6.

Okay now let's use it using our methods from Chapter 1. First we want to factor the top and bottom:
$\lim_{x \to 3} \frac{x-3}{(x+3)(x-3)}$. We will get: $\lim_{x \to 3} \frac{1}{(x+3)}$. Now we can plug in 3 for x and to get 1/6 as the answer.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x \to 0} \frac{1-\cos x}{4x^2}$. Then evaluate it using limit rules that we studied in Chapter 1.

If we plug in 0 for each x, we will get 0/0. This means we can apply L'Hopital's Rule. We will take the derivative of the top and bottom separately resulting in: $\lim_{x \to 0} \frac{\sin x}{8x}$. If we plug in 0 for the top and bottom we will still get 0/0. This means we need to apply L'Hopitals Rule again. We will once again take the derivative of the top and bottom separately: $\lim_{x\to 0}\frac{\cos x}{8}$. This time if we plug in 0 for the top and bottom we get 1/8. Therefore this is our answer.

Okay now let's use it using our methods from Chapter 1. First we want to multiply the top and bottom by the conjugate of the top: $\lim_{x\to 0}\frac{(1-\cos x)}{4x^2}\cdot\left(\frac{1+\cos x}{1+\cos x}\right)$. We will get: $\lim_{x\to 0}\frac{1-\cos^2 x}{4x^2(1+\cos x)}$. We will apply a trig identity on the top: $\lim_{x\to 0}\frac{\sin^2 x}{4x^2(1+\cos x)}$. Now we need to break this up: $\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{x}\cdot\frac{1}{4(1+\cos x)}\right]$. Since $\lim_{x\to 0}\frac{\sin x}{x}=1$, our limit becomes $\left[1\cdot 1\cdot\frac{1}{4(1+\cos 0)}\right]=\left[1\cdot 1\cdot\frac{1}{4(2)}\right]=\frac{1}{8}$.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to 1}\frac{x^3-1}{4x^3-7x+3}$.

If we plug in 1 for each x, we will get 0/0. This means we can apply L'Hopital's Rule. We will take the derivative of the top and bottom separately resulting in: $\lim_{x\to 1}\frac{3x^2}{12x^2-7}$. If we plug in 1 for the top and bottom we will get 3/5. Therefore this is our answer.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to\infty}\frac{x-3x^2}{4x^2+9x}$.

If we plug in ∞ for each x, we will get ∞/∞. This means we can apply L'Hopital's Rule. We will take the first derivative of the top and bottom separately resulting in: $\lim_{x\to\infty}\frac{1-6x}{8x+9}$. If we plug in ∞ for the top and bottom we will still get $-\infty/\infty$ which is still indeterminant. This means we need to apply L'Hopitals Rule again. We will once again take the derivative of the top and bottom separately: $\lim_{x\to\infty}\frac{-6}{8}=-\frac{3}{4}$. This means L'Hopital's Rule no longer applies, so we will stop taking derivatives. The answer is $-3/4$.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to 0}\frac{\sin(3x^2)}{5x}$.

If we plug in 0 for each x, we will get 0/0. This means we can apply L'Hopital's Rule. We will take the first derivative of the top and bottom separately. Remember that you need to apply the chain rule to the numerator: $\lim_{x\to 0}\frac{\cos(3x^2)\cdot 6x}{5}$. If we plug in 0 for the top and bottom we will get $\frac{\cos(0)\cdot 0}{5}=0$ This means L'Hopital's Rule no longer applies, so we will stop taking derivatives. The answer is 0.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to 2}\frac{x-2}{\ln\left(\frac{x}{2}\right)-\sin\frac{\pi x}{2}}$.

If we plug in 2 for each x, we will get 0/0. This means we can apply L'Hopital's Rule. We will take the first derivative of the top and bottom separately. Remember that you need to apply the chain rule to the denominator: $\lim_{x\to 2}\frac{1}{\frac{\frac{1}{2}}{\frac{x}{2}}-\cos\frac{\pi x}{2}\cdot\left(\frac{\pi}{2}\right)}$. We can simplify: $\lim_{x\to 2}\frac{1}{\frac{1}{x}-\cos\frac{\pi x}{2}\cdot\left(\frac{\pi}{2}\right)}$ If we plug in 2 for the top and bottom we will get $\frac{1}{\frac{1}{2}-(-1)\left(\frac{\pi}{2}\right)}$. This simplifies to $\frac{1}{\frac{1+\pi}{2}}=\frac{2}{1+\pi}$ which is our answer.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to 0}\frac{x(1-\cos x)}{3x-\sin 3x}$.

If we plug in 0 for each x, we will get 0/0. This means we can apply L'Hopital's Rule. We will take the first derivative of the top and bottom separately. Remember that you need to apply the chain rule to the denominator. You will need to apply the product rule when taking the derivative of the numerator: $\lim_{x\to 0}\frac{x\cdot\sin x+(1-\cos x)(1)}{3-\cos 3x\cdot 3}$. This simplifies to: $\lim_{x\to 0}\frac{x\sin x+1-\cos x}{3-3\cos 3x}$ If we plug in 0 for the top and bottom we will get $\frac{0\cdot 0+1-1}{3-3}=\frac{0}{0}$. This means L'Hopital's Rule still applies, so we will once again take derivative of the numerator and denominator: $\lim_{x\to 0}\frac{x\cos x+\sin x(1)+\sin x}{3\sin 3x\cdot 3}$. This simplifies to: $\lim_{x\to 0}\frac{x\cos x+2\sin x}{9\sin 3x}$. Plugging in 0 for x will get 0/0 again, so we will again take the derivative of the top and bottom: $\lim_{x\to 0}\frac{x(-\sin x)+\cos x(1)+2\cos x}{9\cos 3x\cdot 3}$. This simplifies to: $\lim_{x\to 0}\frac{-x\sin x+3\cos x}{27\cos 3x}$. Now if we plug in 0 for x on the top and bottom we will get $\lim_{x\to 0}\frac{-x\sin x+3\cos x}{27\cos 3x}=\frac{-0\cdot 0+3(1)}{27(1)}=\frac{3}{27}=\frac{1}{9}$ which is our answer.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to 0}\frac{x\cdot 5^x}{3^x-1}$.

If we plug in 0 for each x, we will get 0/0 (remember any nonzero number raised to the power of zero is 1). This means we can apply L'Hopital's Rule. We will take the first derivative of the top and bottom separately, applying the product rule to the top: $\lim_{x\to 0}\frac{x\cdot 5^x\cdot\ln 5+5^x(1)}{3^x\ln 3}$. If we plug in 0 for the top and bottom we will get $\frac{0\cdot(1)\cdot\ln 5+1}{1\cdot\ln 3}=\frac{1}{\ln 3}$. This means L'Hopital's Rule no longer applies, so our answer is $\frac{1}{\ln 3}$.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to 0}\frac{\sqrt{4x+16}-4}{x}$.

If we plug in 0 for each x, we will get 0/0. This means we can apply L'Hopital's Rule. We can rewrite this as: $\lim_{x\to 0}\frac{(4x+16)^{\frac{1}{2}}-4}{x}$. We will take the first derivative of the top and bottom separately, applying the chain rule to the top: $\lim_{x\to 0}\frac{\frac{1}{2}(4x+16)^{-\frac{1}{2}}(4)}{1}$. This simplifies to: $\lim_{x\to 0}\frac{2}{\sqrt{4x+16}}$ If we plug in 0 for the top and bottom we will get $\frac{2}{\sqrt{16}}=\frac{2}{4}=\frac{1}{2}$. This means L'Hopital's Rule no longer applies, so our answer is $\frac{1}{2}$.

EXAMPLE: Use L'Hopital's Rule to evaluate $\lim_{x\to\infty}\frac{e^{2x}-x^3}{e^{2x}+1}$.

If we plug in ∞ for each x, we will get ∞/∞. This means we can apply L'Hopital's Rule. We will take the first derivative of the top and bottom separately, applying the chain rule: $\lim_{x\to\infty}\frac{2e^{2x}-3x^2}{2e^{2x}}$. If we plug in ∞ for each x, we will still get ∞/∞. This means we can apply L'Hopital's Rule again: $\lim_{x\to\infty}\frac{4e^{2x}-6x}{4e^{2x}}$. Again this gives us ∞/∞, so we will take the derivative of the top and bottom once more: $\lim_{x\to\infty}\frac{8e^{2x}-6}{8e^{2x}}$. This still gives us ∞/∞, so once again we will take derivatives of the top and bottom: $\lim_{x\to\infty}\frac{8e^{2x}}{8e^{2x}}$. Since we have the same thing on the top and the same thing on the bottom, this turns into $\lim_{x\to\infty}1=1$. Therefore the answer to this limit is 1.