Answer Key

Caclulus 120

Worksheet – Optimization Problems

1. Find two numbers whose difference is 150 and whose product is a minimum.

2. A holding pen is being built alongside a long building. The pen requires only three fenced sides, with the building forming the fourth side. There is enough material for 90 m of fencing. Determine the pen's maximum possible area.

3. A box with an open top is to be constructed from a square piece of cardboard, 3 m wide, by cutting out a square from each of the four corners and bending up the sides. Determine the box's largest possible volume.

4. A rectangular page is to contain 120 cm^2 of print. The margins at the top and bottom of the page are 3 cm. The margins on each side are 2.5 cm. Determine the dimensions of the page so that the least amount of paper is used.

5. Find the points on the parabola $y = 4 - x^2$ that is closest to the point $(0, 2)$.

6. A concert promoter is planning the ticket price for an upcoming concert for a certain band. At the last concert, she charge \$70 per ticket and sold 2000 tickets. After conducting a survey, the promoter has determined that for every \$1 decrease in ticket price, she might expect to sell 50 more tickets. What maximum revenue can the promoter expect?

7. A can is to be made to hold 1 L (or 1000 cm^3). Determine the dimensions that will minimize the cost of the metal to make the can.

8. Find two positive numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.

9. An open rectangular box with square base is to be made from 48 ft^2 of material. What dimensions will result in a box with the largest possible volume ?

1. let x and y be the numbers
Difference is 150
Minimize the Product

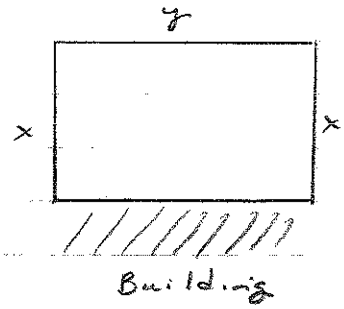
$D = y - x$
$150 = y - x$
$150 + x = y$

$P = xy$
$P = x(150 + x)$
$P = 150x + x^2$
$P' = 150 + 2x$
$0 = 150 + 2x$
$-150 = 2x$
$-75 = x$

$y = 150 + x$
$y = 150 + (-75)$
$y = 75$

The two numbers are 75 and -75.

2. let x represent width $x > 0$
let y represent length $y > 0$
Perimeter: 90 m
Maximize Area



$P = 2x + y$
$90 - 2x = y$ $\quad y > 0, \; 90 - 2x > 0$
$90 > 2x$
$45 > x$
$0 < x < 45$

$A = xy = x(90 - 2x) = 90x - 2x^2$

2. $A = -2x^2 + 90x$

$A' = -4x + 90$

$0 = -4x + 90$

$4x = 90$

$x = 22.5$

$A = -2x^2 + 90x$

$A = -2(22.5)^2 + 90(22.5)$

$A = 1012.5 \text{ m}^2$

$y = 90 - 2x$

$y = 90 - 2(22.5)$

$y = 90 - 45$

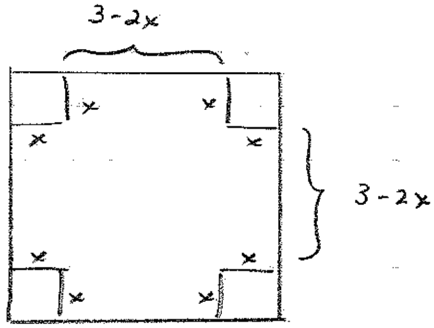
$y = 45$

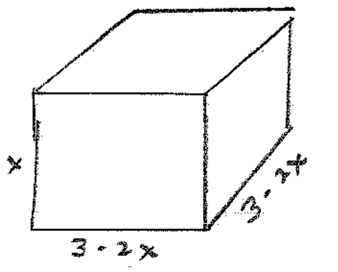
The maximum possible area of the pen is 1012.5 m^2

3. let x represent the width of the corner square. $x > 0$, $3 - 2x > 0$

$3 > 2x$

$1.5 > x$





$V = x(3-2x)^2$

$V = x(9 - 12x + 4x^2)$

$V = 9x - 12x^2 + 4x^3$

$V' = 9 - 24x + 12x^2$

$0 = 9 - 24x + 12x^2$

$0 = 3(4x^2 - 8x + 3)$

$0 = 3[4x^2 - 6x - 2x + 3]$

$0 = 3[2x(2x-3) - (2x-3)]$

$0 = 3(2x-1)(2x-3)$

$x = 0.5 \quad x = 1.5$

↑ Reject

$V = 0.5(3 - 2(0.5))^2$

$V = 0.5(3-1)^2$

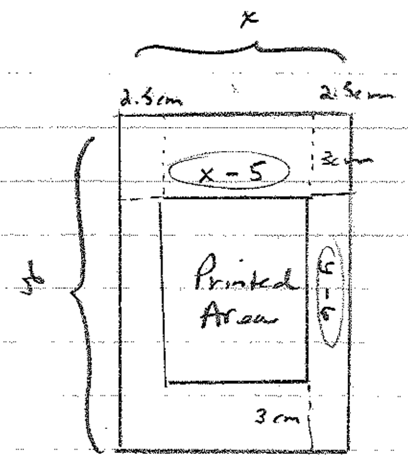
$V = 0.5(2)^2$

$V = 0.5(4)$

$V = 2 \text{ m}^3$

The box's largest possible volume is 2 m^3.

4 Let x represent the width of the page
let y represent the length of the page



Printed area is $120\,cm^2$

$(x-5)(y-6) = 120$

$y - 6 = \dfrac{120}{x-5}$

$y = \dfrac{120}{x-5} + 6$

$x > 5$

$y > 6$

$\dfrac{120}{x-5} + 6 > 0$

$\dfrac{120}{x-5} > 6$

$120 > 6(x-5)$

$120 > 6x - 30$

$120 + 30 > 6x$

$150 > 6x$

$25 > x$ $\quad 0 < x < 25$

Area of the page

$A = xy$

$A = x\left(\dfrac{120}{x-5} + 6\right)$

$A = \dfrac{120x}{x-5} + 6x$

$A' = \dfrac{(x-5)(120) - 120x(1)}{(x-5)^2} + 6$

$A' = \dfrac{\cancel{120x} - 600 - \cancel{120x}}{(x-5)^2} + 6$

$A' = \dfrac{-600}{(x-5)^2} + 6$

$0 = \dfrac{-600}{(x-5)^2} + 6$

$\dfrac{600}{(x-5)^2} = 6$

$600 = 6(x-5)^2$

$100 = (x-5)^2$

4. $(x-5)^2 = 100$

$x-5 = \pm 10$

$x-5 = 10 \qquad x-5 = -10$

$x = 10+5 \qquad x = -10+5$

$x = 15 \qquad x = -5$

$\qquad\qquad \uparrow$ Reject

$y = \dfrac{120}{15-5} + 6$

$y = \dfrac{120}{10} + 6$

$y = 12 + 6$

$y = 18$

If the dimensions of the page are 15 cm by 18 cm, the least amount of paper will be used.

5. $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$d = \sqrt{(x-0)^2 + (y-2)^2}$

$d = \sqrt{x^2 + (y-2)^2}$

$d = \sqrt{4-y + y^2-4y+4}$

$d = \sqrt{y^2-5y+8}$

$d = (y^2-5y+8)^{\frac{1}{2}}$

$d' = \frac{1}{2}(y^2-5y+8)^{-\frac{1}{2}}(2y-5)$

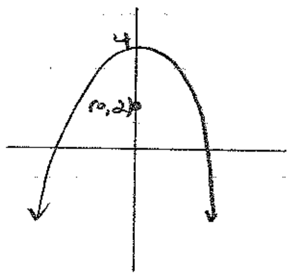
$d' = \dfrac{2y-5}{2(y^2-3y)^{\frac{1}{2}}}$

$0 = 2y-5$

$5 = 2y$

$\frac{5}{2} = y$

$2.5 = y$



$(0,2)$

$y = 4-x^2$

$x^2 = 4-y$

5. $y = 4 - x^2$

$2.5 = 4 - x^2$

$2.5 - 4 = -x^2$

$-1.5 = -x^2$

$1.5 = x^2$

$\pm 1.22 = x$

The points on the parabola closest to (0, 2) are (1.22, 2.5) and (-1.22, 2.5).

6. let x represent the number of times the ticket price is reduced by $1.00.

R = Cost of Ticket × Number of Tickets Sold

$R = (70 - x)(2000 + 50x)$

$R = 140000 + 3500x - 2000x - 50x^2$

$R = 140000 + 1500x - 50x^2$

$R' = 1500 - 100x$

$0 = 1500 - 100x$

$100x = 1500$

$x = \frac{1500}{100}$

$x = 15$

$R = (70 - 15)(2000 + 50(15))$

$R = 55(2750)$

$R = 151250$

The promoter can expect a maximum revenue of $151 250.

7. Let r represent the radius of the can $r > 0$
Let h represent the height of the can $h > 0$

Volume of can $= 1000\ cm^3$

$V = \pi r^2 h$

$1000 = \pi r^2 h$

$\frac{1000}{\pi r^2} = h$

Area of Top πr^2
Area of Bottom πr^2
Area of Cylinder $2\pi r h$

$C = 2\pi r^2 + 2\pi r h$

$C = 2\pi r^2 + 2\cancel{\pi}\cancel{r}\left(\frac{1000}{\cancel{\pi} r^{\cancel{2}}}\right)$

$C = 2\pi r^2 + \frac{2000}{r}$

$C = 2\pi r^2 + 2000 r^{-1}$

$C' = 4\pi r - 2000 r^{-2}$

$C' = 4\pi r - \frac{2000}{r^2}$

$0 = 4\pi r - \frac{2000}{r^2}$

$\frac{2000}{r^2} = 4\pi r$

$2000 = 4\pi r^3$

$\frac{2000}{4\pi} = r^3$

$159.2357 = r^3$

$r = 5.4\ cm$

$h = \frac{1000}{\pi (5.4)^2}$

$h = 10.9\ cm$

The radius and height that will minimize the cost of the can are 5.4 cm and 10.9 cm respectively.

8. Let x and y represent the positive numbers

$x, y > 0$

Sum is 9

Maximize the Product

$S = x + y$

$9 = x + y$

$9 - x = y$ $\quad y > 0,\ 9 - x > 0 \quad 9 > x \quad \therefore\ 0 < x < 9$

$P = x^2(9 - x)$ $\quad$ <u>or</u> $\quad P = (9 - x)^2 x$

$P = 9x^2 - x^3$ $\quad P = (81 - 18x + x^2)x$

$P' = 18x - 3x^2$ $\quad P = 81x - 18x^2 + x^3$

$0 = 18x - 3x^2$ $\quad P' = 81 - 36x + 3x^2$

$0 = -3x(-6 + x)$ $\quad P' = 3(27 - 12x + x^2)$

$x = 0 \quad x = 6$ $\quad P' = 3(x^2 - 12x + 27)$

$\uparrow$ $\quad P' = 3(x - 9)(x - 3)$

reject $\quad x = 9 \quad x = 3$

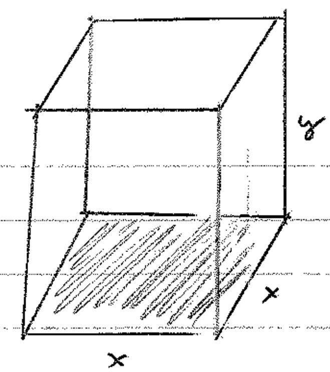
$\uparrow$

reject

$x = 6$ $\quad x = 3$

$y = 9 - 6 = 3$ $\quad y = 9 - 3 = 6$

The two positive numbers are 3 and 6.



9. let x represent the width and length of the box $x > 0$

let y represent the height of the box $y > 0$

Surface Area of the box 48 ft²

Maximize Volume

$SA = x^2 + 4xy$

$48 = x^2 + 4xy$

$48 - x^2 = 4xy$

$\dfrac{48 - x^2}{4x} = y$

$\dfrac{48 - x^2}{4x} > 0$, $0 < x < 6.93$

$48 - x^2 > 0$

$48 > x^2$

$6.93 > x$

$V = x^2 y$

$V = x^2 \left(\dfrac{48 - x^2}{4x}\right)$

$V = x\left(\dfrac{48 - x^2}{4}\right)$

$V = \dfrac{48x - x^3}{4} = \frac{1}{4}(48x - x^3)$

$V' = \frac{1}{4}(48 - 3x^2)$

$0 = \frac{1}{4}(48 - 3x^2)$

$0 = 48 - 3x^2$

$3x^2 = 48$

$x^2 = 16$

$x = \pm 4$

(Reject $x = -4$)

$y = \dfrac{48 - x^2}{4x}$

$y = \dfrac{48 - 4^2}{4(4)}$

$y = \dfrac{48 - 16}{16}$

$y = 2$

$V = x^2 y = 4^2(2) = 16(2) = 32$

The largest possible volume is 32 ft³.